Chapter 6 – Lesson 2: Transforming and Combining Random Variables

Warmup Activity:

Suppose it cost $1 to roll a die and the payoff for each outcome is as follows:

Roll 1 or 2 - Payout is $0.50
Roll 3 - Payout is $3.00
Roll 4, 5, 6 - no payout ($0)

Roll a die 20 times and record each outcome. Calculate your payoff after 20 rolls.

<table>
<thead>
<tr>
<th>Value</th>
<th>Payoff</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.5</td>
<td>2/6</td>
</tr>
<tr>
<td>2</td>
<td>-1.5</td>
<td>1/6</td>
</tr>
<tr>
<td>3</td>
<td>2.00</td>
<td>1/6</td>
</tr>
<tr>
<td>4, 5, 6</td>
<td>-1.00</td>
<td>2/6</td>
</tr>
</tbody>
</table>

What is the probability distribution for this game? Calculate the expected value of the payoff from this game.

\[ E = (-0.5)(2/6) + (2)(1/6) + (-1)(3/6) = -0.33 \]

Is this game worth playing?

Calculator Practice:

Example #1
I conducted an experiment where I tossed a biased coin 5 times and counted the number of heads. After a large number of repetitions of the experiment, I developed the following probability model for the number of heads I would get in 5 tosses of the biased coin.

<table>
<thead>
<tr>
<th>Number of Heads</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.01024</td>
</tr>
<tr>
<td>1</td>
<td>0.0768</td>
</tr>
<tr>
<td>2</td>
<td>0.2304</td>
</tr>
<tr>
<td>3</td>
<td>0.3456</td>
</tr>
<tr>
<td>4</td>
<td>0.2592</td>
</tr>
<tr>
<td>5</td>
<td>0.07776</td>
</tr>
</tbody>
</table>

a) Calculate the expected number of heads that you would observe in 5 tosses of this coin.

\[ \mu = 3 \]  \( \mu \) (mean)  

b) What do the variance and standard deviation of a data distribution describe? (make sure your answer is in context!)

\[ \sigma_x = 1.10 \]

\[ \sigma_x^2 = (1.095)^2 \]

\[ \sigma_x = 1.095 \]  \( \sigma_x \) (standard deviation)
Given the distribution of two independent random variables, \( X \) and \( Y \):

\[
\begin{array}{c|c|c|c}
X & 1 & 2 & 3 \\
\hline
P(X) & 0.2 & 0.5 & 0.3 \\
\end{array}
\quad
\begin{array}{c|c|c}
Y & 2 & 4 \\
\hline
P(Y) & 0.7 & 0.3 \\
\end{array}
\]

\[
\mu_X = 2.1 \quad \sigma_X^2 = 0.49 \quad \mu_Y = 2.6 \quad \sigma_Y^2 = 0.84
\]

**Example #1**
Construct the Probability distribution for \( X + Y \)

\[
\begin{array}{c|c|c|c|c|c}
X + Y & 3 & 4 & 5 & 6 & 7 \\
\hline
P(X + Y) & 0.14 & 0.35 & 0.15 & 0.09 & \text{Sum} \\
\end{array}
\]

\[
\mu_{X+Y} = 4.7 \quad \sigma_{X+Y}^2 = 1.33
\]

**Example #2**
Construct the Probability distribution for \( X - Y \)

\[
\begin{array}{c|c|c|c|c|c}
X - Y & -3 & -2 & -1 & 0 & 1 \\
\hline
P(X - Y) & 0.06 & 0.15 & 0.23 & 0.35 & 0.21 \text{ Sum} \\
\end{array}
\]

\[
\mu_{X-Y} = -0.5 \quad \sigma_{X-Y}^2 = 1.33
\]

Can you see a relationship between the expected values and variation of \( X \) and \( Y \) and the expected value and variation of the combined random variables?

**Variance:**
\[
\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2
\]

**Expected Value:**
\[
\mu_{X+Y} = \mu_X + \mu_Y \quad \mu_{X-Y} = \mu_X - \mu_Y
\]

\[
4.7 = 2.1 + 2.6 \\
-0.5 = 2.1 - 2.6
\]
<table>
<thead>
<tr>
<th>Mean</th>
<th>Multiply/Divide by a Constant (a)</th>
<th>Add/Subtract a Constant (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{X}$</td>
<td>multiply / divide mean</td>
<td>add / subtract</td>
</tr>
<tr>
<td></td>
<td>$M_T = \text{(transferred mean)}$</td>
<td>$M_T = a\bar{X} + b$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Standard Deviation</th>
<th>Multiply s.d</th>
<th>Add/Subtract</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>$\sigma_T =</td>
<td>a</td>
</tr>
</tbody>
</table>

| Shape of a Data Distribution | no change in shape | no change in shape |

**Transfoming and Combining Random Variables**

*Effect of Linear Transformations on a Random Variable*

**Example #1**

A small ferry runs every half hour from one side of large river to the other. The number of cars $X$ on a randomly chosen ferry trip has the probability distribution shown below. Please use calculator for work below.

<table>
<thead>
<tr>
<th>Cars</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.02</td>
<td>0.05</td>
<td>0.08</td>
<td>0.16</td>
<td>0.27</td>
<td>0.42</td>
</tr>
</tbody>
</table>

a) Find the mean and standard deviation of the random variable ($X$).

b) Make a graph of the probability distribution for the random variable ($X$). Describe its shape.

c) The cost of the ferry trip is $5. Make a graph of the probability distribution for the random variable $M$=money collected on a randomly selected ferry trip. Describe its shape.
d) Find and interpret $\mu_M$ and $\sigma_M$.

e) The ferry's company’s expenses are $20 per trip. Define the random variable $Y$ to be the amount of profit (money collected minus expenses) made by the ferry company on a randomly selected trip. Write an equation for $Y$ in terms of $M$.

f) How does the mean of $Y$ relate to the mean of $M$? What is the practical importance of $\mu_Y$?

g) How does the standard deviation of $Y$ relate to the standard deviation of $M$? What information does $\sigma_Y$ give us?

Summary: Linear Transformation of a Random Variable

Combining Random Variables

Rules for combining Random Variables:

Example #2  
A college uses SAT scores as one criterion for admission. Experience has shown that the distribution of SAT scores among its entire population of applicants is:

$\text{SAT Math Score (X)} \quad \mu_x = 625 \quad \sigma_x = 90$

$\text{SAT Verbal Reasoning (Y)} \quad \mu_y = 590 \quad \sigma_y = 100$

What are the mean and standard deviation of the combined Math and Verbal Reasoning scores among the students applying to this college?
Example #3
An electronics store has determined that its sales of big screen TVs has the following distribution:

<table>
<thead>
<tr>
<th>X = # units sold</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>700</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X)</td>
<td>0.2</td>
<td>0.4</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

a) Is X discrete or continuous?

b) Calculate the mean, variance and standard deviation for the probability distribution.

c) Suppose that the store has a cost structure determined by the formula $C = 55X + 250$. Create the probability distribution for the Cost (C). (we are transforming the random variable, X)

<table>
<thead>
<tr>
<th>Cost (C)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P(C)</td>
<td>0.2</td>
<td>0.4</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Suppose the store also sells computers. The sale of computers follows the distribution below:

<table>
<thead>
<tr>
<th>Y = # units sold</th>
<th>200</th>
<th>300</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(Y)</td>
<td>0.1</td>
<td>0.2</td>
<td>0.5</td>
<td>0.2</td>
</tr>
</tbody>
</table>

d) Find the mean, variance and standard deviation for Y.

c) Suppose the profit is given by the formula: $Z = 200X + 350Y$. Find the mean and standard deviation of the store’s profit.

Combining Normal Random Variables:
Example #4
Tom and George are playing in the club golf tournament. Their scores vary as they play the course frequently. Both men’s scores follow a normal distribution. Tom’s score (X) has a mean of 110 with a standard deviation of 10 and George’s score (Y) has a mean of 100 with a standard deviation of 8. If they play independently, what is the probability that Tom will score lower than George and consequently do better in the tournament?

Homework: pg. 378 # 35, 39, 40, 49, 51, 53, 55, 57, 61, Multiple Choice #65-66  (quiz next class)